Overview of Real Analysis (Folland)

Brett Saiki

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These are notes based on Gerald B. Folland's *Real Analysis, Modern Techniques and Their Applications*. For detailed explanations, refer to the actual text.

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0 Prologue

0.1 The Language of Set Theory

The empty set is denoted by $\emptyset$, and the family of all subsets of a set X is denoted by $\mathcal{P}(X)$:

$$\mathcal{P}(X) = \{E : E \subseteq X\}.$$

If $\mathcal{E}$ is a family of sets, we can form the union and intersection of its members:

$$\bigcup_{E\in\mathcal{E}} E = \{x : x \in E \text{ for some } E \in \mathcal{E}\}$$
$$\bigcap_{E\in\mathcal{E}} E = \{x : x \in E \text{ for all } E \in \mathcal{E}\}$$

Usually, it is more convenient to consider indexed families of sets:

$$\mathcal{E} = \{E_\alpha : \alpha \in A\} = \{E_\alpha\}_{\alpha\in A'}$$

in which case the union and intersection are denoted by

$$\bigcup_{\alpha\in A} E_\alpha, \qquad \bigcap_{\alpha\in A} E_\alpha.$$

If $E_\alpha \cap E_\beta = \emptyset$ whenever $\alpha \neq \beta$, the sets E_α are called **disjoint**.

When considering families of sets indexed by $\mathbb{N}$, our notation will be

$$\{E_n\}_{n=1}^\infty \quad \text{or} \quad \{E_n\}_1^\infty,$$

and likewise or unions and intersections. In this case, the notions of **limit superior** and **limit inferior** are sometimes useful:

$$\limsup E_n = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n, \qquad \liminf E_n = \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty E_n.$$

If E and F are sets, we denote their **difference** by $E \setminus F$:

$$E \setminus F = \{x : x \in E \text{ and } x \notin F\},$$

and their **symmetric difference** by $E\Delta F$:

$$E\Delta F = (E \setminus F) \cup (F \setminus E).$$

When it is clearly understood that all sets in questions are subsets of a fixed set X, we define the **complement** E^c of a set E (in X):

$$E^c = X \setminus E.$$

In this situation we have **deMorgan's laws**:

$$\left(\bigcup_{\alpha\in A} E_\alpha\right)^c = \bigcap_{\alpha\in A} E_\alpha^c, \qquad \left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c.$$

If X and Y are sets, their **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A **relation** from X to Y is a subset of $X \times Y$. (If $Y = X$, we speak of a relation on X.) If R is a relation from X to Y, we shall sometimes write xRy to mean that $(x, y) \in R$. The most important types of relations are the following:

- Equivalence relations: An **equivalence relation** on X is a relation R on X such that

$$xRx \text{ for all } x \in X,$$
$$xRy \text{ iff } yRx,$$
$$xRz \text{ whenever } xRy \text{ and } yRz \text{ for some } y.$$

  The **equivalence class** of an element x is $\{y \in X : xRy\}$. X is the disjoint union of these equivalence classes.

- Orderings. See §0.2.

- Mappings. A **mapping** $f : X \to Y$ is a relation R from X to Y with the property that for every $x \in X$ there is a unique $y \in Y$ such that xRy, in which case we writ $y = f(x)$. Mappings are sometimes called **maps** or **functions**;

If $f : X \to Y$ and $g : Y \to Z$ are mappings, we denote by $g \circ f$ their **composition**:

$$g \circ f : X \to Z, \qquad g \circ f(x) = g(f(x)).$$

If $D \subset X$ and $E \subset Y$, we define the **image** of D and the **inverse image** of E under a mapping $f : X \to Y$ by

$$f(D) = \{f(x) : x \in D\}, \qquad f^{-1}(E) = \{x : f(x) \in E\}.$$

It is easily verified that the map $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by the second formula commutes with union, intersections, and complements:

$$f^{-1}\left(\bigcup_{\alpha\in A}\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha), \quad f^{-1}\left(\bigcap_{\alpha\in A}\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha), \quad f^{-1}(E^c) = \left(f^{-1(E)}\right)^c.$$

(The direct image mapping $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ commutes with unions, but in general not with intersections or complements.)

If $f : X \to Y$ is mapping. X is called the **domain** of f and $f(X)$ is called the **range** of f. f is said to be **injective** if $f(x_1) = f(x_2)$ only when $x_1 = x_2$, **surjective** if $f(X) = Y$, and **bijective** if is both injective and surjective. If f is bijective, it has an inverse $f^{-1} : Y \to X$ such that $f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity mappings on X and Y, respectively. If $A \subset X$, we denote by $f \mid A$ the restriction of f to A:

$$(f \mid A) : A \to Y, \qquad (f \mid A)(x) = f(x) \text{ for } x \in A.$$

A **sequence** in a set X is a mapping from $\mathbb{N}$ into X. (We also use the term **finite sequence** to mean a map from $\{1, \ldots, n\}$ into X where $n \in \mathbb{N}$.) If $f : \mathbb{N} \to X$ is sequence and $g : \mathbb{N} \to \mathbb{N}$ satisfies $g(n) < g(m)$ whenver $n < m$, the composition $f \circ g$ is called a **subsequence** of f.

If $\{X_\alpha\}_{\alpha\in A}$ is an indexed family of sets, their **Cartesian product** $\prod_{\alpha\in A} X_\alpha$ is the set of all maps $f : A \to \bigcup_{\alpha\in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for every $\alpha \in A$. If $X = \prod_{\alpha\in A} X_\alpha$ and $\alpha \in A$, we define the αth **projection** or **coordinate map** $\pi_\alpha : X \to X_\alpha$ by $\pi_\alpha(f) = f(\alpha)$. We also frequently write x and x_α instead of f and $f(\alpha)$ and call x_α the αth **coordinate** of x. If the sets X_α are all equal to some fixed set Y, we denote $\prod_{\alpha\in A} X_\alpha$ by Y^A

$$Y^A = \text{the set of all mappings from } A \text{ to } Y.$$

If $A = \{1, \ldots, n\}$, Y^A is denoted by Y^n and may be identified with the set of ordered n-tuples of elements of Y.

0.2 Orderings

A **partial ordering** on a nonempty set X is a relation R on X with the following properties:

(i) if xRy and yRz, then xRz;

(ii) if xRy and yRx, then $x = y$;

(iii) xRx for all x.

If R also satisfies

(iv) if $x, y \in X$, then either xRy or yRx,

then R is called a **linear** (or **total**) ordering. We observe that a partial ordering on X naturally induces a partial ordering on every nonempty subset of X. Two partially ordered sets X and Y are said to be **order isomorphic** if there is a bijection $f : X \to Y$ such that $x_1 \leq x_2$ if and only if $f(x_1) \leq f(x_2)$.

If X is partially ordered by $\leq$, a **maximal** (resp. **minimal**) **element** of X is an element $x \in X$ such that the only $y \in X$ satisfying $x \leq y$ (resp. $x \geq y$) is x itself. Maximal and minimal elements may not exist, and they need not be unique unless the ordering is linear. If $E \subseteq X$, an **upper** (resp. **lower**) **bound** for E is an element $x \in X$ such that $y \leq x$ (resp. $x \leq y$) for all $y \in E$. An upper bound for E need not be an element of E, and unless E is linearly ordered, a maximal element of E need not be an upper bound for E.

If X is linear ordered by $\leq$ and every nonempty subset of X has a (necessarily unique) minimal element, X is said to be **well ordered** by $\leq$, and $\leq$ is called a **well ordering** on X. For example, $\mathbb{N}$ is well ordered by its natural ordering.

0.1 Proposition (The Hausdorff Maximal Principle)**.** Every partially ordered set has a maximal linearly ordered subset.

0.2 Lemma (Zorn's Lemma)**.** If X is a partially ordered set and every linearly ordered subset of X has an upper bound, then X has a maximal element.

0.3 Theorem (The Well Ordering Principle)**.** Every nonempty set X can be well ordered.

0.4 Theorem (The Axiom of Choice)**.** If $\{X_\alpha\}_{\alpha \in A}$ is a nonempty collection of nonempty sets, then $\prod_{\alpha \in A} X_\alpha$ is nonempty.

0.5 Corollary. If $\{X_\alpha\}_{\alpha \in A}$ is a disjoint colletion of nonempty sets, there is a set $Y \subset_{\alpha \in A} X_\alpha$ such that $Y \cap X_\alpha$ contains precisely one element for each $\alpha \in A$.

0.3 Cardinality

If X and Y are nonempty sets, we define the expressions

$$\text{card}(X) \leq \text{card}(Y), \qquad \text{card}(X) = \text{card}(Y), \qquad \text{card}(X) \geq \text{card}(Y)$$

to mean that exists $f : X \to Y$ which is injective, bijective, or surjective, respectively. We also define

$$\text{card}(X) < \text{card}(Y), \qquad \text{card}(X) > \text{card}(Y)$$

to mean that there is an injection but no bijection, or a surjection but no bijection from X to Y. These relationships can be extended to the empty set by declaring that

$$\text{card}(\emptyset) < \text{card}(X) \text{ and } \text{card}(X) > \text{card}(\emptyset) \text{ for all } X \neq \emptyset.$$

For the remainder of this section, we assume implicitly that all sets in question are nonempty in order to avoid special arguments for $\emptyset$.

0.6 Proposition. $\text{card}(X) \le \text{card}(Y)$ if and only if $\text{card}(Y) \ge \text{card}(X)$.

0.7 Proposition. For any sets X and Y, either $\text{card}(X) \le \text{card}(Y)$ or $\text{card}(Y) \le \text{card}(X)$.

0.8 Theorem (The Schröder-Bernstein Theorem)**.** If $\text{card}(X) \le \text{card}(Y)$ and $\text{card}(Y) \le \text{card}(X)$, then $\text{card}(X) = \text{card}(Y)$.

0.9 Proposition. For any set X, $\text{card}(X) < \text{card}(\mathcal{P}(X))$.

A set X is called **countable** (or **denumerable**) if $\text{card}(X) \le \text{card}(\mathbb{N})$. In particular, all finite sets are countable, and for these it is convenient to interpret "$\text{card}(X)$" as the number of elements in X:

$$\text{card}(X) = n \Longleftrightarrow \text{card}(X) = \text{card}(\{1, \ldots, n\}).$$

If X is countable but not finite, we say that X is **countably infinite**.

0.10 Proposition.

(i) If X and Y are countable, so is $X \times Y$.

(ii) If A is countable and X_α is countable for every $\alpha \in A$, then $\bigcup_{\alpha \in A} X_\alpha$ is countable.

(iii) If X is countably infinite, then $\text{card}(X) = \text{card}(\mathbb{N})$.

0.11 Corollary. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

A set X is said to have the **cardinality of the continuum** if $\text{card}(X) = \text{card}(\mathbb{R})$. We shall use the letter $\mathfrak{c}$ as an abbreviation for $\text{card}(\mathbb{R})$:

$$\text{card}(X) = \mathfrak{c} \Longleftrightarrow \text{card}(X) = \text{card}(\mathbb{R}).$$

0.12 Proposition. $\text{card}(\mathcal{P}(\mathbb{N})) = \mathfrak{c}$.

0.13 Corollary. If $\text{card}(X) \ge \mathfrak{c}$, then X is uncountable.

0.14 Proposition.

(i) If $\text{card}(X) \le \mathfrak{c}$ and $\text{card}(Y) \le \mathfrak{c}$, then $\text{card}(X \times Y) \le \mathfrak{c}$.

(ii) If $\text{card}(A) \le \mathfrak{c}$ and $\text{card}(X_\alpha) \le \mathfrak{c}$ for all $\alpha \in A$, then $\text{card}\left(\bigcup_{\alpha \in A}\right) \le \mathfrak{c}$.

0.4 More about Well Ordered Sets

Let X be a well ordered set. If $A \subseteq X$ is nonempty, A has a minimal element, which is its maximal lower bound or **infimum**; we shall denote it by $\inf A$. If A is bounded above, it also has a minimal upper bound or **supremum**, denoted by $\sup A$. If $x \in X$, we define the **initial segment** of x to be

$$I_x = \{y \in X : y < x\}.$$

The elements of I_x are called **predecessors** of x.

0.15 Proposition (The Principle of Transfinite Induction)**.** Let X be a well ordered set. If A is a subset such that $x \in A$ whenever $I_x \subset A$, then $A = X$.

0.16 Proposition. If X is a well ordered and $A \subset X$, then $\bigcup_{x \in A} I_x$ is either an initial segment or X itself.

0.17 Proposition. If X and Y are well ordered, then either X is order isomorphic to Y, or X is order isomorphic to an initial segment in Y, or Y is order isomorphic to an initial segment in X.

0.18 Proposition. There is an uncountable well ordered set Ω such that I_x is countable for each $x \in \Omega$. If Ω' is another set with the same properties, then Ω and Ω' are order isomorphic.

The set Ω in Proposition 0.18, which is essentially unique qua well ordered set, is called the **set of countable ordinals**.

0.19 Proposition. Every countable subset of Ω has an upper bound.

It is sometimes convenient to add an extra element ω_1 to Ω to form a set $\Omega^* = \Omega \cup \{\omega_1\}$ and to extend the ordering on Ω to Ω^* by declaring that $x < \omega_1$ for all $x \in \Omega$. ω_1 is called the **first uncountable ordinal**.

0.5 The Extended Real Number System

It is frequently useful to adjoin two extra points ∞ and $-\infty$ to $\mathbb{R}$ to form the **extended real number system** $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and to extend the usual ordering on $\mathbb{R}$ by declaring that $-\infty < x < \infty$ for all $x \in \mathbb{R}$. The completeness of $\mathbb{R}$ can be stated as follows: every subset A of $\overline{\mathbb{R}}$ has a least upper bound, or **supremum**, and a greatest lower bound, or **infinimum**, which are denoted by $\sup A$ and $\inf A$. If $A = \{a_1, \ldots, a_n\}$, we also write

$$\max(a_1, \ldots, a_n) = \sup A, \qquad \min(a_1, \ldots, a_n) = \inf A.$$

From completeness it follows that every sequence $\{x_n\}$ in $\overline{\mathbb{R}}$ has a **limit superior** and a **limit inferior**:

$$\limsup x_n = \inf_{k \geq 1} \left(\sup_{n \geq k} x_n \right), \qquad \liminf x_n = \sup_{k \geq 1} \left(\inf_{n \geq k} x_n \right).$$

The sequence $\{x_n\}$ converges (in $\mathbb{R}$) if and only if these two numbers are equal (and finite), in which case its limit is their common values. One can also define lim sup and lim inf for functions $f : \mathbb{R} \to \overline{\mathbb{R}}$ for instance:

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \left(\sup_{0 < |x-a| < \delta} f(x) \right).$$

The arithmetical operations on $\mathbb{R}$ can be partially extended to $\overline{\mathbb{R}}$:

$$x \pm \infty = \pm\infty (x \in \mathbb{R}), \qquad \infty + \infty = \infty, \qquad -\infty - \infty = -\infty,$$
$$x \cdot (\pm\infty) = \infty (x > 0), \qquad x \cdot (\pm\infty) = \mp\infty (x < 0).$$

We make no attempt to define $\infty - \infty$, but we abide by the convention that, unless otherwise stated,

$$0 \cdot (\pm\infty) = 0.$$

We employ the following notation for intervals in $\overline{\mathbb{R}}$: if $-\infty \leq a < b \leq \infty$,

$$(a, b) = \{x : a < x < b\}, \qquad [a, b] = \{x : a \leq x \leq b\},$$
$$(a, b] = \{x : a < x \leq b\}, \qquad [a, b) = \{x : a \leq x < b\}$$

We shall occasionally encounter uncountable sums of nonnegative numbers. If X is an arbitrary set and $f : X \to [0, \infty]$, we define $\sum_{x \in X} f(x)$ to be supremum of its finite partial sums:

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subset X, F \text{ finite} \right\}.$$

0.20 Proposition. Given $f : X \to [0, \infty]$, let $A = \{x : f(x) > 0\}$. If A is uncountable, then $\sum_{x \in X} f(x) = \infty$. If A is countably infinite, then $\sum_{x \in X} f(x) = \sum_1^\infty f(g(n))$ where $g : \mathbb{N} \to A$ is any bijection and the sum on the right is an ordinary infinite series.

Some terminology concerning (extended) real-valued functions: A relation between numbers that is applied to functions is understood to hold pointwise. Thus $f \leq g$ means that $f(x) \leq g(x)$ for every x, and $\max(f, g)$ is the function whose value at x is $\max(f(x), g(x))$. If $X \subset \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$, f is called **increasing** if $f(x) \leq f(y)$ whenver $x \leq y$ and **strictly increasing** if $f(x) < f(y)$ whenever $x < y$; similarly for **decreasing**. A function that is either increasing or decreasing is called **monotone**.

If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then f has right- and left-hand limits at each point:

$$f(a+) = \lim_{x \searrow a} f(x) = \inf_{x > a} f(x), \qquad f(a-) = \lim_{x \nearrow a} f(x) = \sup_{x < a} f(x).$$

Moreover, the limiting values $f(\infty) = \sup_{a \in \mathbb{R}} f(x)$ and $f(-\infty) = \inf_{a \in \mathbb{R}} f(x)$ exist (possibly equal to $\pm\infty$). f is called **right continuous** if $f(a) = f(a+)$ for all $a \in \mathbb{R}$ and **left continuous** if $f(a) = f(a-)$ for all $a \in \mathbb{R}$.

For points x in $\mathbb{R}$ or $\mathbb{C}$, $|x|$ denotes the ordinary absolute value or modulus of x, $|a + ib| = \sqrt{a^2 + b^2}$. For points x in $\mathbb{R}^n$ or $\mathbb{C}^n$, $|x|$ denotes the Euclidean norm:

$$|x| = \left[\sum_1^n |x_j|^2 \right]^{1/2}.$$

We recall that a set $U \subset \mathbb{R}$ is **open** if, for every $x \in U$, U includes an interval centered at x.

0.21 Proposition. Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

0.6 Metric Spaces

A **metric** on a set X is a function $\rho : X \times X \to [0, \infty)$ such that

- $\rho(x, y) = 0$ if and only if $x = y$;
- $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

(Intuitively, $\rho(x, y)$ is to be interpreted as the distance from x to y.) A set equipped with a metric is called a **metric space**. Some examples:

(i) The Euclidean distance $\rho(x, y) = |x - y|$ is a metric on $\mathbb{R}^n$.

(ii) $\rho_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$ and $\rho_\infty(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$ are metrics on the space of continuous functions on $[0, 1]$.

(iii) If ρ is a metric on X and $A \subset X$, then $\rho \mid (A \times A)$ is a metric on A.

(iv) If (X_1, ρ_1) and (X_2, ρ_2) are metric spaces, the **product metric** ρ on $X_1 \times X_2$ is given by

$$\rho((x_1, x_2), (y_1, y_2)) = \max(\rho_1(x_1, y_1), \rho_2(x_2, y_2)).$$

Let (X, ρ) be a metric space. If $x \in X$ and $r > 0$, the (open) **ball** of radius r about x is

$$B(r, x) = \{y \in X : \rho(x, y) < r\}.$$

A set $E \subset X$ is **open** if for every $x \in E$ there exists $r > 0$ such that $B(r, x) \subset E$, and **closed** if its complement is open. For example, every ball $B(r, x)$ is open. Also, X and $\emptyset$ are both open and closed. The union of any collection of open sets is open, and hence the intersection of any collection of closed sets is closed. Also, the intersection (resp. union) of any finite collection of open (resp. closed) sets is open (resp. closed).

If $E \subset X$, the union of all open sets $U \subset E$ is the largest open set contained in E, it is called the **interior** of E and is denoted by E^o. Likewise, the intersection of all closed sets $F \supset E$ is the smallest closed set containing E; it is called the **closure** of E and is denoted by $\overline{E}$. E is said to be **dense** in X if $\overline{E} = X$, and **nowhere dense** if $\overline{E}$ has empty interior. X is called **separable** if it has a countable dense subset. A sequence $\{x_n\}$ in X converges to $x \in X$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$.

0.22 Proposition. If X is a metric space, $E \subset X$, and let $x \in X$, the following are equivalent:
(i) $x \in \overline{E}$;
(ii) $B(r, x) \cap E \neq \emptyset$ for all $r > 0$;
(iii) There is a sequence $\{x_n\}$ in E that converges to x.

If (X_1, ρ_1) and (X_2, ρ_2) are metric spaces, a map $f : X_1 \to X_2$ is called **continuous** at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho_2(f(y), f(x)) < \epsilon$ whenever $\rho_1(x, y) < \delta$ — in other words, such that $f^{-1}(B(\epsilon, f(x))) \supset B(\delta, x)$. The map f is called **continuous** if it is continuous as each $x \in X_1$ and **uniformly continuous** if, in addition, the δ in the definition of continuity can be chosen independent of x.

0.23 Proposition. $f : X_1 \to X_2$ is continuous if and only if $f^{-1}(U)$ is open in X_1 for every open $U \subset X_2$.

A sequence $\{x_n\}$ in a metric space (X, ρ) is called **Cauchy** if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. A subset E of X is called **complete** if every Cauchy sequence in E converges and its limit is in E.

0.24 Proposition. A closed subset of a complete metric space is complete, and the complete subset of an arbitrary metric space is closed.

In a metric space (X, ρ) we can define the distance from a point to a set and the distance between two sets. Namely, if $x \in X$ and E, $F \subset X$,

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\},$$
$$\rho(E, F) = \inf\{\rho(x, y) : x \in E, y \in F\} = \inf\{\rho(x, F) : x \in E\}.$$

Observe that, by Proposition 0.22, $\rho(x, E) = 0$ if and only if $x \in \overline{E}$. We also define the **diameter** of $E \subset X$ to be

$$\operatorname{diam} E = \sup\{\rho(x, y) : x, y \in E\}.$$

E is called **bounded** if $\operatorname{diam} E < \infty$.

If $E \subset X$ and $\{V_\alpha\}_{\alpha \in A}$ is a collection of sets such that $E \subset \bigcup_{\alpha \in A} V_\alpha$, $\{V_\alpha\}_{\alpha \in A}$ is called a **cover** of E, and E is said to be **covered** by the V_α's. E is called totally bounded if, for every $\epsilon > 0$, E can be covered by finitely many balls of radius ϵ. Every totally bounded set is bounded. (The converse is false in general.) If E is totally bounded, so is $\overline{E}$, for it is easily seen that if $E \subset \bigcup_1^n B(\epsilon, z_j)$, then $\overline{E} \subset \bigcup_1^n B(2\epsilon, z_j)$.

0.25 Theorem. If E is a subset of the metric space (X, ρ), the following are equivalent:
(i) E is complete and totally bounded.
(ii) (**The Bolzano-Weierstrass Property**) Every sequence in E has a subsequence that converges to a point of E.
(iii) (**The Heine-Borel Property**) If $\{V_\alpha\}_{\alpha \in A}$ is a cover of E by open sets, there is a finite set $F \subset A$ such that $\{V_\alpha\}_{\alpha \in F}$ covers E.

A set E that posses the properties (i)-(iii) of Theorem 0.25 is called **compact**. Every compact set is closed (by Proposition 0.22) and bounded; the converse is false in general but true in $\mathbb{R}^n$.

0.26 Proposition. Every closed and bounded subset of $\mathbb{R}^n$ is compact.

Two metrics ρ_1 and ρ_2 on a set X are called **equivalent** if

$$C\rho_1 \leq \rho_2 \leq C'\rho_1 \text{ for some } C, C' > 0.$$

It is easily verified that equivalent metrics define the same open, closed, and compact sets, the same convergent and Cauchy sequences, and the same continuous and uniformly conntinuous mappins. Consequently, most results concerning metric spaces depend not on the particular metric chosen but only on its equivalence class.

1 Measures

1.1 Introduction

This section was skipped.

1.2 σ-Algebras

Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions of complements; in other words, if $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_1^n E_j \in \mathcal{A}$; and if $E \in \mathcal{A}$, then $E^c = \mathcal{A}$. A **σ-algebra** is an algebra that is closed under countable unions.

We observe that since $\bigcap_j E_j = \left(\bigcup_j E_j^c\right)^c$, algebras (resp. σ-algebras) are also closed under finite (resp. countable) intersections. Moreover, if $\mathcal{A}$ is an algebra, then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$, for if $E \in \mathcal{A}$ we have $\emptyset = E \bigcap E^c$ and $X = E \bigcup E^c$.

It is worth noting that an algebra $\mathcal{A}$ is an σ-algebra provided that is is closed under countable disjoint unions. Indeed, suppose $\{E_j\}_1^\infty \subset \mathcal{A}$. Set

$$F_k = E_k \setminus \left[\bigcup_1^{k-1} E_j\right] = E_k \cap \left[\bigcup_1^{k-1} E_j\right]^c.$$

Then the F_k's belong to $\mathcal{A}$ and are disjoint, and $\bigcup_1^\infty E_j = \bigcup_1^\infty F_k$.

Some examples: If X is any set, $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are σ-algebras. If X is uncountable, then

$$\mathcal{A} = \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}$$

is a σ-algebra called the **σ-algebra of countable or co-countable sets**. It is trivial to verify that the intersection of any family of σ-algebras on X is again a σ-algebra. It follows that if $\mathcal{E}$ is any subset of $\mathcal{P}(X)$, there is a unique smallest σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$. (There is always at least one such, namely, $\mathcal{P}(X)$.) $\mathcal{M}(\mathcal{E})$ is called that σ-algebra **generated** by $\mathcal{E}$.

1.1 Lemma. If $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.

If X is any metric space, or more generally any topological space, the σ-algebra generated by the collection of open sets in X is called the **Borel σ-algebra** on X and is denoted by $\mathcal{B}_X$. Its members are called **Borel sets**. $\mathcal{B}_X$ thus includes open sets, closed sets, countable intersections of open sets, countable unions of closed sets, and so forth.

1.2 Proposition. $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:

(i) the open intervals: $\mathcal{E}_1 = \{(a,b) : a < b\}$,

(ii) the closed intervals: $\mathcal{E}_2 = \{[a,b] : a < b\}$,

(iii) the half-open intervals: $\mathcal{E}_3 = \{(a,b] : a < b\}$ or $\mathcal{E}_4 = \{[a,b) : a < b\}$,

(iv) the open rays: $\mathcal{E}_5 = \{(a,\infty] : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) : a \in \mathbb{R}\}$,

(v) the closed rays: $\mathcal{E}_7 = \{[a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] : a \in \mathbb{R}\}$.

Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the **product σ-algebra** on X is the σ-algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$. (If $A = \{1,\ldots,n\}$ we also write $\bigoplus_1^n \mathcal{M}_j$ or $\mathcal{M}_1 \bigoplus \cdots \bigoplus \mathcal{M}_n$.)

1.3 Proposition. If A is countable, then $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is the σ-algebra generated by $\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$.

1.4 Proposition. Suppose that $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$, $\alpha \in A$. Then $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\}$. If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all α, $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\mathcal{F}_2 = \{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$.

1.5 Proposition. Let $X_1,\ldots,X_n$ be metric spaces and let $X = \prod_1^n X_j$, equipped with the product metric. Then $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. If the X_j's are separable, then $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$.

1.6 Corollary. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$.

We define an **elementary family** to be a collection $\mathcal{E}$ of subsets of X such that

(i) $\emptyset \in \mathcal{E}$;

(ii) if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$;

(iii) if $E \in \mathcal{E}$, then E^c is a finite disjoint union of members of $\mathcal{E}$.

1.7 Proposition. If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.

1.3 Measures

Let X be a set equipped with a σ-algebra $\mathcal{M}$. A **measure** on $\mathcal{M}$ (or on $(X,\mathcal{M})$, or simply on X if $\mathcal{M}$ is understood) is a function $\mu : M \to [0,\infty]$ such that

(i) $\mu(\emptyset) = 0$,

(ii) if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \mu(E_j)$.

Property (ii) is called **countable additivity**. It implies **finite additivity**:

(ii′) if $E_1,\ldots,E_n$ are disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^n E_j) = \sum_1^n \mu(E_j)$.

because one can take $E_j = \emptyset$ for $j > n$. A function μ that satisfies (i) and (ii′) but not necessarily (ii) is called a **finitely additive measure**.

If X is a set $\mathcal{M} \subset \mathcal{P}(X)$ is σ-algebra, $(X,\mathcal{M})$ is called a **measurable space** and the sets in $\mathcal{M}$ are called **measurable sets**. If μ is a measure on $(X,\mathcal{M})$, then $(X,\mathcal{M},\mu)$ is called a **measure space**.

Let $(X,\mathcal{M},\mu)$ be a measure space. If $\mu(X) < \infty$, μ is called **finite**. If $X = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, μ is called **σ-finite**. More generally, if $E = \bigcup_1^\infty E_j$ where $E_j \in M$ and $\mu(E_j) < \infty$ for all j, the set E is said to be **σ-finite** for μ. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, μ is called **semifinite**. Every σ-finite measure is semifinite but not conversely.

Let us examine a few examples of measures.

- Let X be any nonempty set, $\mathcal{M} = \mathcal{P}(X)$, and f any function from X to $[0, \infty]$. Then f determines a measure μ on $\mathcal{M}$ by the function $\mu(E) = \sum_{x \in E} f(x)$. The reader may verify that μ is semifinite if and only if $f(x) < \infty$ for every $x \in X$, and μ is σ-finite if and only if μ is semifinite and $\{x : f(x) > 0\}$ is countable. Two special cases are of particular significance: If $f(x) = 1$ for all x, μ is called **counting measure**; and if, for some $x_0 \in X$, f is defined by $f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$, μ is called the **point mass** or **Dirac measure** at x_0.

- Let X be an uncountable set, and let $\mathcal{M}$ be the σ-algebra of countable or co-countable sets. The function μ on $\mathcal{M}$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is co-countable is easily seen to be a measure.

- Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$. Define $\mu(E) = 0$ if E is finite, $\mu(E) = \infty$ if E is infinite. Then μ is a finitely additive measure but not a measure.

1.8 Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) (**Monotonicity**) If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.

(ii) (**Subadditivity**) If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.

(iii) (**Continuity from below**) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j \to \infty} \mu(E_j)$.

(iv) (**Continuity from above**) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \supset E_2 \supset \cdots$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j \to \infty} \mu(E_j)$.

If $(X, \mathcal{M}, \mu)$ is a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called **null** set. By subadditivity, any countable union of null sets is a null set. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true **almost everywhere**, or for **almost every** x. (If more precision is needed, we shall speak of a μ**-null** set, or μ**-almost everywhere**).

If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ by monotonicity provided that $F \in \mathcal{M}$, but in general it need not be true that $F \in \mathcal{M}$. A measure whose domain includes all subsets of null sets is called **complete**.

1.9 Theorem. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E\mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

The measure $\overline{\mu}$ in Theorem 1.9 is called the **completion** of μ, and $\overline{\mathcal{M}}$ is called the **completion** of $\mathcal{M}$ with respect to μ.

1.4 Outer Measures

An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

(i) $\mu^*(\emptyset) = 0$,

(ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$,

(iii) $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$.

1.10 Proposition. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \mu(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j \right\}.$$

Then μ^* is an outer measure.

If μ^* is an outer measure on X, a set $A \cap X$ is called **μ^*-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X.$$

The inequality $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ holds for any A and E, so to prove that A is μ^*-measurable, it suffices to prove the reverse inequality.

1.11 Theorem (Carathéodory's Theorem)**.** If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ will be called a **premeasure** if

(i) $\mu_0(\emptyset) = 0$,

(ii) if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$.

In particular, a premeasure is finitely addivite since one can take $A_j = \emptyset$ for j large. The notions of finite and σ-finite premeasures are defined just as for measures. If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, it induces an outer measure on X in accordance with Proposition 1.10, namely,

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \mu^*(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j \right\}. \tag{1.12}$$

1.13 Proposition. If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by (1.12), then

(i) $\mu^* \mid \mathcal{A} = \mu_0$;

(ii) every set in $\mathcal{A}$ is μ^* measureable.

1.14 Theorem. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 — namely, $\mu = \mu^* \mid \mathcal{M}$ where μ^* is given by (1.12). If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality where $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

1.5 Borel Measures on the Real Line

We begin with a more general construction that yields a large family of measures on $\mathbb{R}$ whose domain is the Borel σ-algebra $\mathcal{B}_\mathbb{R}$; such measures are called **Borel measures** on $\mathbb{R}$. To motivate the ideas, suppose μ is a finite Borel meausre on $\mathbb{R}$, and let $F(x) = \mu((-\infty . x_n])$. ($F$ is somestimes called the **distribution function** of μ.) Then F is increasing by Theorem 1.8(a) and right continuous by Theorem 1.8(d) since $(-\infty, x] = \bigcap_1^\infty (-\infty, x_n]$ whenever $x_n \searrow x$. Our procedure will be to turn this process around and construcgt a measure μ starting from an increasing, right-countinuous function F. The special case $F(x) = x$ will yield the usual "length" measure.

1.15 Proposition. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j] (j = 1, \ldots, n)$ are disjoint half-open intervals, let

$$\mu_0 \left(\bigcup_1^n (a_j, b_j] \right) = \sum_1^n [F(b_j) - F(a_j)].$$

and let $mu_0(\emptyset) = 0$. Then μ_0 is a premeasure on the algebra $\mathcal{A}$.

1.16 Theorem. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all a, b. If G is another such function, we have $\mu_F = \mu_G$ if and only if $F - G$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0,x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu((-x,0]) & \text{if } x < 0, \end{cases}$$

then F is increasing and right continuous, and $\mu = \mu_F$.

From the theory of §1.4, for each increasing and right continuous F, we have not only the Borel measure μ_F, but a complete measures $\overline{\mu}_F$ whose domain includes $\mathcal{B}_{\mathbb{R}}$. In fact, $\overline{\mu}_F$ is just the completion of μ_F and one can show that its domain is always strictly larger than $\mathcal{B}_{\mathbb{R}}$. We shall usually denote this complete measure also by μ_F; is is called the **Lebesgue-Stieltjes measure** associated to F.

1.17 Lemma. For any $E \in \mathcal{M}_\mu$,

$$\mu(E) = \inf\{\sum_1^\infty \mu((a_j,b_j)) : E \subset \bigcup_1^\infty (a_j,b_j)\}.$$

1.18 Theorem. If $E \in \mathcal{M}_\mu$, then

$$\begin{aligned} \mu(E) &= \inf\{\mu(U) : U \supset E \text{ and } U \text{ is open}\} \\ &= \sup\{\mu(U) : K \subset E \text{ and } K \text{ is compact}\}. \end{aligned}$$

1.19 Theorem. If $E \subset \mathbb{R}$, the following are equivalent.

(i) $E \in \mathcal{M}_\mu$.

(ii) $E = V \setminus N_1$ where V is a G_δ set and $\mu(N_1) = 0$.

(iii) $E = H \cap N_2$ where H is a F_δ set and $\mu(N_2) = 0$.